Birational invariants defined by Lawson homology

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Abstract

New birational invariants for a projective manifold are defined by using Lawson homology. These invariants can be highly nontrivial even for projective threefolds. Our techniques involve the weak factorization theorem of Wlodarczyk and tools developed by Friedlander, Lawson, Lima-Filho and others. A blowup formula for Lawson homology is given in a separate section. As an application, we show that for each $n \geq 5$, there is a smooth rational variety X of dimension n such that the Griffiths groups $\mathrm{Griff}_p(X)$ are infinitely generated even modulo torsion for all p with $2 \leq p \leq n-3$.

1 Introduction

In this paper, all varieties are defined over $\mathbb{C}$. Let X be an n-dimensional projective variety. The **Lawson homology** $L_pH_k(X)$ of p-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F], [L1]). For general background, the reader is referred to [L2].

In [FM], Friedlander and Mazur showed that there are natural transformations, called **cycle class maps**

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X).$$

Define

$$L_pH_k(X)_{hom} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}.$$

The **Griffiths group** of codimension q-cycles is defined to

$$\mathrm{Griff}^q(X) := \mathcal{Z}^q(X)_{hom}/\mathcal{Z}^q(X)_{alg}$$

It was proved by Friedlander [F] that, for any smooth projective variety X, $L_pH_{2p}(X) \cong \mathcal{Z}_p(X)/\mathcal{Z}_p(X)_{alg}$. Therefore

$$L_pH_{2p}(X)_{hom} \cong \mathrm{Griff}_p(X),$$

where $\mathrm{Griff}_p(X) := \mathrm{Griff}^{n-p}(X)$.

The main result in this paper is the following

Theorem 1.1 *If X is a smooth n-dimensional projective variety, then $L_1H_k(X)_{hom}$ and $L_{n-2}H_k(X)_{hom}$ are smooth birational invariants for X. More precisely, if $\varphi : X \to X'$ is a birational map between smooth projective manifolds X and X', then φ induces isomorphisms $L_1H_k(X)_{hom} \cong L_1H_k(X')_{hom}$ for $k \geq 2$ and $L_{n-2}H_k(X)_{hom} \cong L_{n-2}H_k(X')_{hom}$ for $k \geq 2(n-2)$. In particular, $L_1H_k(X)_{hom} = 0$ and $L_{n-2}H_k(X)_{hom} = 0$ for any smooth rational variety.*

Corollary 1.1 *Let X be a smooth rational projective variety with* $\dim(X) \leq 4$, *then $\Phi_{p,k} : L_pH_k(X) \to H_k(X)$ is injective for all $k \geq 2p \geq 0$.*

Remark 1.1 *In general, for $2 \leq p \leq n-3$, $L_pH_k(X)_{hom}$ is* not *a birational invariant for the smooth projective variety X. This follows from the blowup formula in Lawson homology (See Corollary 1.2, 1.3).*

Remark 1.2 *If $p = 0, n-1, n$, then $L_pH_k(X)_{hom} = 0$ for all $k \geq 2p$. In these cases, the statement in the theorem is trivial. The case for $p = 0$ follows from Dold-Thom theorem ([DT]). The case for $p = n-1$ is due to Friedlander [F]. The case for $p = n$ is from the definition. In particular, these invariants are trivial for smooth projective varieties with dimension less than or equal to two.*

Another result is this paper is the following:

Theorem 1.2 *(Lawson homology for a blowup) Let X be smooth projective manifold and $Y \subset X$ be a smooth subvariety of codimension r. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \to \tilde{X}_Y$ the exceptional divisor of the blowing up. Then for each p, k with $k \geq 2p \geq 0$, we have the following isomorphism*

$$I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \bigoplus L_pH_k(X) \cong L_pH_k(\tilde{X}_Y)$$

As applications, we have the following

Corollary 1.2 *For each $n \geq 5$, there exists a* rational *manifold X with $\dim(X) = n$ such that*

$$\dim_{\mathbb{Q}}\{\mathrm{Griff}_p(X) \otimes \mathbb{Q}\} = \infty, \quad 2 \leq p \leq n-3.$$

Corollary 1.3 *For any integer $p > 1$ and $k \geq 0$, there exists* rational *projective manifold X such that $L_pH_{k+2p}(X) \otimes \mathbb{Q}$ is an infinite dimensional vector space over $\mathbb{Q}$.*

The main tools used to prove the main result are: the long exact localization sequence given by Lima-Filho in [Li], the explicit formula for the Lawson homology of codimension-one cycles on a smooth projective manifold given by Friedlander in [F], and the weak factorization theorem proved by Wlodarczyk and others in [W] and in [AKMW].

2 Some fundamental materials in Lawson homology

First recall that for a morphism $f : U \to V$ between projective varieties, there exist induced homomorphism

$$f_* : L_pH_k(U) \to L_pH_k(V)$$

for all $k \geq 2p \geq 0$, and if $g : V \to W$ is another morphism between projective varieties, then

$$(g \circ f)_* = g_* \circ f_*.$$

Furthermore, it has been shown by C. Peters [Pe] that if U and V are smooth and projective, there are Gysin "wrong way" homomorphisms $f^* : L_pH_k(V) \to L_{p-c}H_{k-2c}(U)$, where $c = \dim(V) - \dim(U)$. If $g : V \to W$ is another morphism between smooth projective varieties, then

$$(g \circ f)^* = f^* \circ g^*.$$

Recall also the fact that there is a long exact sequence (cf. [Li], also [FG])

$$\cdots \to L_pH_k(U - V) \to L_pH_k(U) \to L_pH_k(V) \to L_pH_{k-1}(U - V) \to \cdots,$$

where U is quasi-projective and $U-V$ is any algebraic closed subset in U.

Let X be a smooth projective variety and $i_0 : Y \hookrightarrow X$ a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowup. Set $U := X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and j the inclusion $U \subset \tilde{X}_Y$. Note that $\pi : D = \sigma^{-1}(Y) \to Y$ makes D into a projective bundle of rank $r-1$, given precisely by $D = \mathrm{P}(N_{Y/X})$ and we have (cf. [[V], pg. 271])

$$\mathcal{O}_{\tilde{X}_Y}(D)|_D = \mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1).$$

Denote by h the class of $\mathcal{O}_{\mathrm{P}(N_{Y/X})}(-1)$ in $\mathrm{Pic}(D)$. We have $h = -D_{|D}$ and $-h = i^*i_* : L_qH_m(D) \to L_{q-1}H_{m-2}(D)$ for $0 \leq 2q \leq m$ ([FG], Theorem 2.4], [[Pe], Lemma 11]). The last equality can be equivalently regarded as a Lefschetz operator

$$-h = i^*i_* : L_qH_m(D) \to L_{q-1}H_{m-2}(D), \quad 0 \leq 2q \leq m. \tag{1}$$

The proof of the main result is based on the following lemmas:

Lemma 2.1 *For each $p \geq 0$, we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(D) & \xrightarrow{i_*} & L_pH_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_pH_k(U) & \xrightarrow{\delta_*} & L_pH_{k-1}(D) & \to \cdots \\
 & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & \\
\cdots \to & L_pH_k(Y) & \xrightarrow{(i_0)^*} & L_pH_k(X) & \xrightarrow{j_0^*} & L_pH_k(U) & \xrightarrow{(\delta_0)_*} & L_pH_{k-1}(Y) & \to \cdots
\end{array}$$

Proof. This is from the corresponding commutative diagram of fibration sequences of p-cycles. More precisely, to show the first square, we begin from the following commutative diagram

$$\begin{array}{ccc}
D & \xhookrightarrow{i} & \tilde{X}_Y \\
\downarrow \pi & & \downarrow \sigma \\
Y & \xhookrightarrow{i_0} & X.
\end{array}$$

From this, we obtain the corresponding commutative diagram of p-cycles:

$$\begin{array}{ccc}
\mathcal{Z}_p(D) & \xhookrightarrow{i_*} & \mathcal{Z}_p(\tilde{X}_Y) \\
\downarrow \pi_* & & \downarrow \sigma_* \\
\mathcal{Z}_p(Y) & \xhookrightarrow{(i_0)_*} & \mathcal{Z}_p(X).
\end{array}$$

Since Y is a smooth projective variety, $\tilde{X}_Y$ and D are smooth projective varieties, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_p(\tilde{X}_Y) & \to & \mathcal{Z}_p(\tilde{X}_Y)/\mathcal{Z}_p(D) \\
\downarrow \sigma_* & & \downarrow\cong \\
\mathcal{Z}_p(X) & \to & \mathcal{Z}_p(X)/\mathcal{Z}_p(Y).
\end{array}$$

Therefore we obtain the following commutative diagram of the fibration sequences of p-cycles

$$\begin{array}{ccccc} \mathcal{Z}_p(D) & \overset{i_*}{\hookrightarrow} & \mathcal{Z}_p(\tilde{X}_Y) & \to & \mathcal{Z}_p(\tilde{X}_Y)/\mathcal{Z}_p(D) \\ \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong \\ \mathcal{Z}_p(Y) & \overset{(i_0)_*}{\hookrightarrow} & \mathcal{Z}_p(X) & \to & \mathcal{Z}_p(X)/\mathcal{Z}_p(Y). \end{array}$$

where the fact that the rows are fibration sequences is due to Lima- Filho [Li].

By taking the homotopy groups of these fibration sequences, we get the long exact sequences of commutative diagram given in the Lemma.

□

Proposition 2.1 *If $p = 0$, then we have the following commutative diagram*

$$\begin{array}{cccccccccc} \cdots \to & H_k(D) & \overset{i_*}{\to} & H_k(\tilde{X}_Y) & \overset{j^*}{\to} & H_k^{BM}(U) & \overset{\delta_*}{\to} & H_{k-1}(D) & \to & \cdots \\ & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & & \\ \cdots \to & H_k(Y) & \overset{(i_0)_*}{\to} & H_k(X) & \overset{j_0^*}{\to} & H_k^{BM}(U) & \overset{(\delta_0)_*}{\to} & H_{k-1}(Y) & \to & \cdots \end{array}$$

Moreover, if $x \in H_k(D)$ maps to zero under π_* and i_*, then $x = 0 \in H_k(D)$.

Proof. The first conclusion follows directly from Lemma 2.1 with $p = 0$ and the Dold-Thom Theorem. For the second conclusion assume $i_*(x) = 0$ and $\pi_*(x) = 0$. Then there exists an element $y \in H_{k+1}^{BM}(U)$ such that the image of y under the boundary map $(\delta_0)_* : H_{k+1}^{BM}(U) \to H_k(Y)$ is 0 by the given condition. Hence there exists an element $z \in H_{k+1}(X)$ such that $(j_0)^*(z) = y$. Now the surjectivity of the map $\sigma_* : H_{k+1}(\tilde{X}_Y) \to H_{k+1}(X)$ implies that there is an element $\tilde{z} \in H_{k+1}(\tilde{X}_Y)$ such that $j^*(\tilde{z}) = y$. Therefore, $x = 0 \in H_k(D)$.

□

Corollary 2.1 *If $p = n-2$, then we have the following commutative diagram*

$$\begin{array}{cccccccccc} \cdots \to & L_{n-2}H_k(D) & \overset{i_*}{\to} & L_{n-2}H_k(\tilde{X}_Y) & \overset{j^*}{\to} & L_{n-2}H_k(U) & \overset{\delta_*}{\to} & L_{n-2}H_{k-1}(D) & \to & \cdots \\ & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & & \\ \cdots \to & L_{n-2}H_k(Y) & \overset{(i_0)_*}{\to} & L_{n-2}H_k(X) & \overset{j_0^*}{\to} & L_{n-2}H_k(U) & \overset{(\delta_0)_*}{\to} & L_{n-2}H_{k-1}(Y) & \to & \cdots \end{array}$$

Lemma 2.2 *For each p, we have the following commutative diagram*

$$\begin{array}{cccccccccc} \cdots \to & L_pH_k(D) & \overset{i_*}{\to} & L_pH_k(\tilde{X}_Y) & \overset{j^*}{\to} & L_pH_k(U) & \overset{\delta_*}{\to} & L_pH_{k-1}(D) & \to & \cdots \\ & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & & \\ \cdots \to & H_k(D) & \overset{i_*}{\to} & H_k(\tilde{X}_Y) & \overset{j^*}{\to} & H_k^{BM}(U) & \overset{\delta_*}{\to} & H_{k-1}(D) & \to & \cdots \end{array}$$

In particular, it is true for $p = 1, n-2$.

Proof. See [Li] and also [FM]. □

Lemma 2.3 *For each p, we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(Y) & \stackrel{(i_0)_*}{\to} & L_pH_k(X) & \stackrel{j^*}{\to} & L_pH_k(U) & \stackrel{(\delta_0)_*}{\to} & L_pH_{k-1}(Y) & \to \cdots \\
 & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
\cdots \to & H_k(Y) & \stackrel{(i_0)_*}{\to} & H_k(X) & \stackrel{j^*}{\to} & H_k^{BM}(U) & \stackrel{(\delta_0)_*}{\to} & H_{k-1}(Y) & \to \cdots
\end{array}$$

In particular, it is true for $p = 1, n-2$.

Proof. See [Li] and also [FM]. □

3 Lawson homology for blowups

As an application of Lemma 2.1, we give an explicit formula for a blowup in Lawson homology. Since it may have some independent interest, we devote a separate section to it. First, we want to revise the projective bundle theorem given by Friedlander and Gabber ([FG], Prop.2.5). It is convenient to extend the definition of Lawson homology by setting

$$L_pH_k(X) = L_0H_k(X), \quad if \quad p < 0.$$

Now we have the following revised "Projective Bundle Theorem":

Proposition 3.1 *Let E be an algebraic vector bundle of rank r over a smooth projective variety Y, then for each $p \geq 0$ we have*

$$L_pH_k(\mathrm{P}(E)) \cong \bigoplus_{j=0}^{r-1} L_{p-j}H_{k-2j}(Y)$$

where $\mathrm{P}(E)$ is the projectivization of the vector bundle E.

Remark 3.1 *The difference between this and the projective bundle theorem of [FG] is that here we place no restriction on p.*

Proof. For $p \geq r-1$, this is exactly the projective bundle theorem given in [FG]. If $p < r-1$, we have the same method of [FG], i.e., the localization sequence and the naturality of Φ, to reduce to the case in which E is trivial. From

$$\mathcal{Z}_0(\mathrm{P}^{r-1} \times Y) \to \mathcal{Z}_0(\mathrm{P}^r \times Y) \to \mathcal{Z}_0(\mathbb{C}^r \times Y),$$

we have the long exact localization sequence given at the beginning of section 2:

$$\cdots \to L_0H_k(\mathrm{P}^{r-1} \times Y) \to L_0H_k(\mathrm{P}^r \times Y) \to L_0H_k(\mathbb{C}^r \times Y) \to L_0H_{k-1}(\mathrm{P}^{r-1} \times Y) \to \cdots .$$

From this, and the Künneth formula for $\mathrm{P}^r \times Y$, we have the following isomorphism:

$$(*) \quad H_{k-2r}(Y) \cong L_0 H_k(\mathbb{C}^r \times Y) \cong H_k^{BM}(\mathbb{C}^r \times Y).$$

Note that

$$(**) \qquad H_{k-2r}(Y) \cong L_{p-r} H_{k-2r}(Y) \quad \text{if} \quad p \le r.$$

All the remaining arguments are the same as those in [[FG], Prop 2.5], as we review in the following.

We want to use induction on r. For $r-1=p$, the conclusion holds. From the commutative diagram of abelian groups of cycles:

$$\begin{array}{ccc}
\{\oplus_{j=0}^{p} Z_{p-j}(X)\} \bigoplus \{\oplus_{j=p+1}^{r-1} Z_0(X \times \mathbb{C}^{j-p})\} & \rightarrow & \{\oplus_{j=0}^{p} Z_{p-j}(X)\} \bigoplus \{\oplus_{j=p+1}^{r} Z_0(X \times \mathbb{C}^{j-p})\} \\
\downarrow & & \downarrow \\
Z_p(X \times \mathrm{P}^{r-1}) & \rightarrow & Z_p(X \times \mathrm{P}^r)
\end{array}$$

We obtain the commutative diagram of fibration sequences:

$$\begin{array}{ccccc}
\{\oplus_{j=0}^{p} Z_{p-j}(X)\} \bigoplus \{\oplus_{j=p+1}^{r-1} Z_{p-j}(X)\} & \rightarrow & \{\oplus_{j=0}^{p} Z_{p-j}(X)\} \bigoplus \{\oplus_{j=p+1}^{r} Z_{p-j}(X)\} & \rightarrow & Z_0(X \times \mathbb{C}^{r-p}) \\
\downarrow & & \downarrow & & \downarrow \\
Z_p(X \times \mathrm{P}^{r-1}) & \rightarrow & Z_p(X \times \mathrm{P}^r) & \rightarrow & Z_p(X \times \mathbb{C}^r)
\end{array}$$

where $Z_{p-j}(X) := Z_0(X \times \mathbb{C}^{j-p})$ for $p - j < 0$.

The first vertical arrow is a homotopy equivalence by induction. The last one is a homotopy equivalence by Complex Suspension Theorem [L1]. Hence by the Five Lemma, we obtain the homotopy equivalence of the middle one.

The proof is completed by combining this with (*) and (**) above.

$\square$

Remark 3.2 *The isomorphism*

$$\psi : \bigoplus_{j=0}^{r-1} L_{p-j} H_{k-2j}(Y) \xrightarrow{\cong} L_p H_k(\mathrm{P}(E))$$

in Proposition 3.1 is given explicitly by

$$\psi(u_0, u_1, \cdots, u_{r-1}) = \sum_{j=0}^{r-1} h^j \pi^* u_j$$

where h is the Lefschetz hyperplane operator $h : L_q H_m(\mathrm{P}(E)) \to L_{q-1} H_{m-2}(\mathrm{P}(E))$ defined in (1). For $p \ge r-1$, this explicit formula has been proved in [[FG], Prop. 2.5]. In the remaining cases, h is the Lefschetz hyperplane operator $h : H_m(\mathrm{P}(E)) \to H_{m-2}(\mathrm{P}(E))$ defined in (1).

Using the notations in section 2, we have the following:

Theorem 3.1 *(Lawson homology for a blowup) Let X be smooth projective manifold and $Y \subset X$ be a smooth subvariety of codimension r. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the natural map, and $i : D = \sigma^{-1}(Y) \to \tilde{X}_Y$ the exceptional divisor of the blowing up. Then for each p, k with $k \geq 2p \geq 0$, we have the following isomorphism*

$$I_{p,k} : \left\{ \bigoplus_{1\leq j\leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \xrightarrow{\cong} L_pH_k(\tilde{X}_Y)$$

given by

$$I_{p,k}(u_1, \cdots, u_{r-1}, u) = \sum_{j=1}^{r-1} i_* h^j \pi^* u_j + \sigma^* u$$

Proof. Let $U := \tilde{X}_Y - D = X - Y$. By the definitions of the maps i, π and σ, and Lemma 2.1, we have the following commutative diagram of the long exact localization sequences:

$$\begin{array}{ccccccccccc} \cdots \to & L_pH_k(D) & \xrightarrow{i_*} & L_pH_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_pH_k(U) & \xrightarrow{\delta_*} & L_pH_{k-1}(D) & \to & \cdots \\ & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & & \\ \cdots \to & L_pH_k(Y) & \xrightarrow{(i_0)_*} & L_pH_k(X) & \xrightarrow{j_0^*} & L_pH_k(U) & \xrightarrow{(\delta_0)_*} & L_pH_{k-1}(Y) & \to & \cdots \end{array} \tag{2}$$

From this, and the surjectivity of j^*, we have

$$L_pH_{2p}(\tilde{X}_Y) = \sigma^* L_pH_{2p}(X) + i_* L_pH_{2p}(D).$$

By the "revised" projective bundle theorem above, for any $p \geq 0$, there is an isomorphism

$$L_pH_k(D) \cong \bigoplus_{j=0}^{r-1} h^j \pi^* L_{p-j}H_{k-2j}(Y), \quad 0 \leq 2p \leq k.$$

Hence we see that

$$L_pH_{2p}(\tilde{X}_Y) = \sigma^* L_pH_{2p}(X) + \Sigma_{j=0}^{r-1} i_* h^j \pi^* L_{p-j}H_{2p-2j}(Y). \tag{3}$$

But clearly by Lemma 2.1 and the projective bundle theorem, if $u \in L_pH_k(Y)$, then

$$\sigma_*(i_* h^{r-1} \pi^*(u)) = (i_0)_*(u).$$

Since σ is a birational morphism, it has degree one. As a directly corollary of the projection formula (cf. [Pe], Lemma 11 c.), we have $\sigma_*(\sigma^* a) = a$ for any $a \in L_pH_k(X)$. We have

$$\sigma_*(\sigma^*((i_0)_* u)) = (i_0)_* u, \quad u \in L_pH_k(Y).$$

Thus we obtain the relations

$$v := i_* h^{r-1} \pi^* u - \sigma^*((i_0)_* u) \in \ker\, \sigma_*, \quad u \in L_p H_k(Y)$$

Since $j^* = (j_0)^* \sigma_*$ in (2), we get $j^*(v) = 0$. From the exactness of the upper row in (2), we get

$$v \in \sum_{j=1}^{r-1} i_* h^j L_{p-j} H_{k-2j}(Y). \tag{4}$$

The equality (3) and the relation (4) together imply immediately that the map $I_{p,2p}$ is surjective for the case $k = 2p$.

To prove the injectivity for the case that $k = 2p$, consider $(u_1, u_2, \cdots, u_{r-1}, u) \in \ker\, I_{p,2p}$. Applying σ_*, we find that $u = 0$. Note that $i^* i_* = -h$. Applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1} \pi^* u_j = 0 \in L_{p-1} H_{k-2}(D).$$

The isomorphism in Proposition 3.1 implies that $u_j = 0$ for $1 \le j \le r-1$. This completes the proof for the case $k = 2p$.

From this and (2), we have

$$\begin{array}{ccccccccc}
\cdots \to & L_p H_{2p+1}(D) & \overset{i_*}{\to} & L_p H_{2p+1}(\tilde{X}_Y) & \overset{j^*}{\to} & L_p H_{2p+1}(U) & \overset{\delta_*}{\to} & 0 \\
& \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \\
\cdots \to & L_p H_{2p+1}(Y) & \overset{(i_0)_*}{\to} & L_p H_{2p+1}(X) & \overset{j_0^*}{\to} & L_p H_{2p+1}(U) & \overset{(\delta_0)_*}{\to} & 0
\end{array} \tag{5}$$

Now the situation for $k = 2p+1$ is the same as that in the case $k = 2p$. From (5) and the "revised" projective bundle theorem, we have

$$L_p H_{2p+1}(\tilde{X}_Y) = \sigma^* L_p H_{2p+1}(X) + \Sigma_{j=0}^{r-1} i_* h^j \pi^* L_{p-j} H_{2p+1-2j}(Y). \tag{6}$$

From (4) and (6), we obtain the surjectivity of $I_{p,2p+1}$ for the case that $k = 2p+1$.

To prove the injectivity, consider $(u_1, u_2, \cdots, u_{r-1}, u) \in \ker\, I_{p,2p+1}$. Applying σ_*, we find that $u = 0$. Note that $i^* i_* = -h$. By applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1}\pi^* u_j = 0 \in L_{p-1}H_{k-2}(D).$$

The isomorphism in Proposition 3.1 again implies that $u_j = 0$ for $1 \le j \le r-1$. This completes the proof for the case $k = 2p+1$.

Now for $k \geq 2p+2$, we reach the same situation as those in the case that $k = 2p$ or $k = 2p+1$. More precisely, we give the complete argument by using mathematical induction.

Suppose that we have

$$\begin{array}{ccccccccc} \cdots \to & L_pH_{2p+m}(D) & \xrightarrow{i_*} & L_pH_{2p+m}(\tilde{X}_Y) & \xrightarrow{j^*} & L_pH_{2p+m}(U) & \xrightarrow{\delta_*} & 0 \\ & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \\ \cdots \to & L_pH_{2p+m}(Y) & \xrightarrow{(i_0)_*} & L_pH_{2p+m}(X) & \xrightarrow{j_0^*} & L_pH_{2p+m}(U) & \xrightarrow{(\delta_0)_*} & 0 \end{array} \tag{7}$$

for some integer $m \geq 0$.

We want to prove that $I_{p,2p+m}$ is an isomorphism and

$$\begin{array}{ccccccccc} \cdots \to & L_pH_{2p+m+1}(D) & \xrightarrow{i_*} & L_pH_{2p+m+1}(\tilde{X}_Y) & \xrightarrow{j^*} & L_pH_{2p+m+1}(U) & \xrightarrow{\delta_*} & 0 \\ & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \\ \cdots \to & L_pH_{2p+m+1}(Y) & \xrightarrow{(i_0)_*} & L_pH_{2p+m+1}(X) & \xrightarrow{j_0^*} & L_pH_{2p+m+1}(U) & \xrightarrow{(\delta_0)_*} & 0 \end{array} \tag{8}$$

Once this step is done, it completes the proof of the theorem.

From the assumption (7), we have

$$L_pH_{2p+m}(\tilde{X}_Y) = \sigma^* L_pH_{2p+m}(X) + \Sigma_{j=0}^{r-1} i_* h^j \pi^* L_{p-j}H_{2p+m-2j}(Y). \tag{9}$$

From (4) for $k = 2p+m$ and (9), we obtain the surjectivity of $I_{p,2p+m}$ for the case that $k = 2p+m$.

To prove the injectivity, consider $(u_1, u_2, \cdots, u_{r-1}, u) \in \ker I_{p,2p+m}$. Applying σ_*, we find that $u = 0$. Note that $i^* i_* = -h$. By applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1}\pi^* u_j = 0 \in L_{p-1}H_{k-2}(D).$$

The isomorphism in Proposition 3.1 once again implies that $u_j = 0$ for $1 \le j \le r-1$. This completes the proof for the case $k = 2p+m$. Now (7) automatically reduces to (8) and this completes the proof of the theorem. $\square$

As an application, this result gives many examples of smooth projective manifolds (even rational ones) for which the Griffiths group of p-cycles is infinitely generated (even modulo torsion) for $p \geq 2$. Recall that the Griffiths group $\mathrm{Griff}_p(X)$ is defined to be the p-cycles homologically equivalent to zero modulo the subgroup of p-cycles algebraically equivalent to zero.

Example: Note the fact in [F] that $\mathrm{Griff}_2(\tilde{X}_Y) \cong L_2H_4(\tilde{X}_Y)_{hom}$. For $X = \mathrm{P}^5$, $Y \subset \mathrm{P}^4$ the general hypersurface of degree 5, we obtain an infinite dimensional $\mathbb{Q}$-vector space $\mathrm{Griff}_2(\tilde{X}_Y) \otimes \mathbb{Q}$ from the fact $\dim_{\mathbb{Q}}(\mathrm{Griff}_1(Y) \otimes \mathbb{Q}) = \infty$ (cf. [C]). It gives the example mentioned in Remark 1.1.

From the blowup formula for Lawson homology and Clemens' result [C], we have the following

Corollary 3.1 *For each $n \geq 5$, there exists a rational manifold X with $\dim(X) = n$ such that*

$$\dim_{\mathbb{Q}} \mathrm{Griff}_p(X) \otimes \mathbb{Q} = \infty, \quad 2 \leq p \leq n-3.$$

Proof. Note that $\mathrm{Griff}_p(X) \cong L_pH_{2p}(X)_{hom}$ for any smooth projective variety X. Now the remaining argument is the direct result of Theorem 3.1 and the above result of Clemens [C].

□

More generally, from the blowup formula for Lawson homology and a result given by the author [H], we have the following

Corollary 3.2 *For any integers $p > 1$ and $k \geq 0$, there exists a* rational *projective manifold X such that $L_pH_{k+2p}(X) \otimes \mathbb{Q}$ is infinite dimensional vector space over $\mathbb{Q}$.*

Proof. It follows from the blowup formula for Lawson homology and Theorem 1.4 in [H]. For example, if $p = 2$, $k = 1$, we can find a rational projective manifold X with $\dim(X) = 6$ such that $L_2H_5(X) \otimes \mathbb{Q}$ is infinite dimensional $\mathbb{Q}$-vector space.

□

4 The proof of the main theorem

The following result will be used several times in the proof of our main theorem:

Theorem 4.1 *(Friedlander [F]) Let X be any smooth projective variety of dimension n. Then we have the following isomorphisms*

$$\begin{cases} L_{n-1}H_{2n}(X) \cong \mathbf{Z}, \\ L_{n-1}H_{2n-1}(X) \cong H_{2n-1}(X, \mathbf{Z}), \\ L_{n-1}H_{2n-2}(X) \cong H_{n-1,n-1}(X, \mathbf{Z}) = NS(X) \\ L_{n-1}H_k(X) = 0 \quad for \quad k > 2n. \end{cases}$$

□

Remark 4.1 *In the following, we adopt the notational convention $H_k(X) = H_k(X, \mathbf{Z})$.*

Now we begin the proof of our main results. There are two parts of the proof of the main theorem: $p = 1$ and $p = n - 2$.

Proof of the main theorem ($p = 1$):

Case A: $\sigma_* : L_1H_k(\tilde{X}_Y)_{hom} \to L_1H_k(X)_{hom}$ is injective.

We will use the commutative diagrams in Lemma 2.1–2.3.

Let $a \in L_1H_k(\tilde{X}_Y)_{hom}$ be such that $\sigma_*(a) = 0$. By Lemma 2.1, we have $j^*(a) = 0 \in L_1H_k(U)$ and hence there exists an element $b \in L_1H_k(D)$ such that $i_*(b) = a$. Set $\tilde{b} = \pi_*(b)$. By the commutative diagram in Lemma 2.1 again, we have $(i_0)_*(\tilde{b}) = 0 \in L_1H_k(X)$. By the exactness of the rows in the commutative diagram, there exists an element $\tilde{c} \in L_1H_{k+1}(U)$ such that the image of $\tilde{c}$ under the boundary map $(\delta_0)_* : L_1H_{k+1}(U) \to L_1H_k(Y)$ is $\tilde{b}$. Note that δ_* is the other boundary map $\delta_* : L_1H_{k+1}(U) \to L_1H_k(D)$. Therefore, $\pi_*(b - \delta_*(\tilde{c})) = 0 \in L_1H_k(Y)$ and $j_*(b - \delta_*(\tilde{c})) = a$. Now by the "revised" Projective Bundle Theorem and Dold-Thom theorem ([DT]), we have $L_1H_k(D) \cong L_1H_k(Y) \oplus L_0H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots \cong L_1H_k(Y) \oplus H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots$. We know $b - \delta_*(\tilde{c}) \in H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots$. By the explicit formula of the cohomology (and homology) for a blowup ([GH]), we know each map $H_{k-2*}(Y) \to H_k(\tilde{X}_Y)$ is injective. Hence a must be zero in $L_1H_k(\tilde{X}_Y)$. This is the injectivity of σ_*.

Case B: $\sigma_* : L_1H_k(\tilde{X}_Y)_{hom} \to L_1H_k(X)_{hom}$ is surjective.

Let $a \in L_1H_k(X)_{hom}$. From the surjectivity of the map $\sigma_* : L_1H_k(\tilde{X}_Y) \to L_1H_k(X)$, there exists an element $\tilde{a} \in L_1H_k(\tilde{X}_Y)$ such that $\sigma_*(\tilde{a}) = a$. Set $\tilde{b} = \Phi_{1,k}(\tilde{a})$. By the commutative diagram in Lemma 2.1, we have $j^*(\tilde{b}) = 0 \in H_k^{BM}(U)$. From the exactness of the rows of the diagram in Lemma 2.1, we have an element $\tilde{c} \in H_k(D)$ such that $i_*(\tilde{c}) = \tilde{b}$. Set $c = \pi_*(\tilde{c})$. Then $(i_0)_*(c) = 0$ by the assumption of a and the commutative of the diagram in Lemma 2.1. Using the exactness of rows in Lemma 2.1 again, we can find an element $d \in H_{k+1}^{BM}(U)$ such that $(\delta_0)_*(d) = c$. Hence $i_*(\tilde{c} - \delta_*(d)) = \tilde{b} \in H_k(\tilde{X}_Y)$ and $\pi_*(\tilde{c} - \delta_*(d)) = 0$. Now we need to use the formula $L_1H_k(D) \cong L_1H_k(Y) \oplus H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots$ again. From this we can find an element $e \in L_1H_k(D)$ such that $\Phi_{1,k}(e) = \tilde{c} - \delta(d)$. Obviously, $\Phi_{1,k}(\tilde{a} - i_*(e)) = 0$ and $\sigma_*(\tilde{a} - i_*(e)) = a$ as we want.

□

Proof of the main theorem ($p = n - 2$):

Case 1: σ_* is injective.

The injectivity of $j_0^* : L_{n-2}H_k(X)_{hom} \to L_{n-2}H_k(U)_{hom}$ is trivial since the $\dim(Y) \le n - 2$, where $j_0 : U \to X$ is the inclusion. In fact, if $\dim(Y) < n - 2$, $j_0^* : L_{n-2}H_k(X) \to L_{n-2}H_k(U)$ is an isomorphism and so is $j_0^* : L_{n-2}H_k(X)_{hom} \to L_{n-2}H_k(U)_{hom}$. If $\dim(Y) = n - 2$, then for $k \ge 2(n-2) + 1$ the injectivity of j_0^* is from the commutative diagram in Lemma 2.2, and the vanishing of $L_{n-2}H_k(Y)$ and $H_k(Y)$; for $k = 2(n-2)$,

the injectivity of j_0^* is from the commutative diagram in Lemma 2.2, and the nontriviality of $(i_0)_* : H_{2(n-2)}(Y) \to H_{2(n-2)}(X)$, since Y is a Kähler submanifold of X with complex dimension $n-2$.

Now we need to prove $j^* : L_{n-2}H_k(\tilde{X}_Y)_{hom} \to L_{n-2}H_k(U)_{hom}$ is injective, where $j : U \to \tilde{X}_Y$ the inclusion. Let $a \in L_{n-2}H_k(\tilde{X}_Y)_{hom}$ such that $j^*(a) = 0 \in L_{n-2}H_k(U)_{hom}$, then there exists an element $b \in L_{n-2}H_k(D)$ such that $i_*(b) = a$. Now by the commutative diagram in Corollary 2.1, we have $j_0^*(\sigma_*(a)) = 0$. Set $a' \equiv \sigma_*(a)$. From the exactness of localization sequence in the bottom row in Corollary 2.1, there is an element $b' \in L_{n-2}H_k(Y)$ such that $(i_0)_*(b') = a'$.

Claim: In the commutative diagram in Corollary 2.1, there exists an element $c' \in L_{n-2}H_{k+1}(U)$ such that $(\delta_0)_*(c') = b'$ under the map $(\delta_0)_* : L_{n-2}H_{k+1}(U) \to L_{n-2}H_k(Y)$ and $\delta_*(c') = b$ under the map $\delta_* : L_{n-2}H_{k+1}(U) \to L_{n-2}H_k(D)$.

Proof of the claim: Since $\Phi_{n-2,k} : L_{n-2}H_k(Y) \cong H_k(Y)$ (note: $k \geq 2(n-2) \geq \dim(Y)$), we use the same notation b' for its image in $H_k(Y)$ since $L_{n-2}H_k(Y) \to H_k(Y)$ is injective for all $k \geq 2(n-2)$. At the beginning of the proof of the injectivity of the main theorem, we have already shown that $j_0^* : L_{n-2}H_k(X)_{hom} \to L_{n-2}H_k(U)_{hom}$ is injective. That is to say, $(i_0)_*(b') = 0 \in L_{n-2}H_k(X)_{hom}$. Hence there exists an element $c \in L_{n-2}H_{k+1}(U)$ such that whose image is b' under the boundary map $(\delta_0)_* : L_{n-2}H_{k+1}(U) \to L_{n-2}H_k(Y)$. Let $\tilde{b}$ be the image of c under the map $L_{n-2}H_{k+1}(U) \to L_{n-2}H_k(D)$. Now $\pi_*(\tilde{b} - b) = 0 \in L_{n-2}H_k(Y)$ and $i_*(\Phi_{n-2,k}(\tilde{b} - b)) = 0 \in H_k(\tilde{X}_Y)$, by Proposition 2.1, we have $\Phi_{n-2,k}(\tilde{b} - b) = 0$. Since $\Phi_{n-2,k}$ is injective on $L_{n-2}H_k(D)$ (see Theorem 4.1), we get $\tilde{b} - b = 0$. This c satisfies both conditions of the claim.

□

Now everything is clear. The element a comes from the element c in $L_{n-2}H_{k+1}(U)$. By the exactness of the localization sequence in the upper row in Lemma 2.1, we get $a = 0 \in L_{n-2}H_k(\tilde{X}_Y)$. This completes the proof of the injectivity.

Case 2: σ_* is surjective.

Similar to the injectivity, the surjectivity of $j_0^* : L_{n-2}H_k(X)_{hom} \to L_{n-2}H_k(U)_{hom}$ is trivial since the $\dim(Y) \leq n-2$, where $j_0 : U \to X$ is the inclusion. In fact, if $\dim(Y) < n-2$, $j_0^* : L_{n-2}H_k(X) \to L_{n-2}H_k(U)$ is an isomorphism and so is $j_0^* : L_{n-2}H_k(X)_{hom} \to L_{n-2}H_k(U)_{hom}$. If $\dim(Y) = n-2$, then the surjectivity of j_0^* is from the commutative diagram in Lemma 2.3, and the isomorphism $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(Y) \cong H_{2(n-2)}(Y) \cong \mathbf{Z}$.

We only need to show $j^* : L_{n-2}H_k(\tilde{X}_Y)_{hom} \cong L_{n-2}H_k(U)_{hom}$, where $j : U \to \tilde{X}_Y$ the inclusion. There are a few cases.

(a) For the case that $k = 2(n-2)$, the map $j^* : L_{n-2}H_k(\tilde{X}_Y) \to L_{n-2}H_k(U)$ is a surjective map. Hence the induced map j^* on $L_{n-2}H_k(\tilde{X}_Y)_{hom}$ is also surjective by trivial reason.

(b) The case that $k = 2(n-2)+1$. By the commutative diagram in Lemma 2.2, and note that the map $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(D) \to H_{2(n-2)}(D)$ is injective, we have, for $a \in L_{n-2}H_{2(n-2)+1}(U)_{hom}$, the image of a under the boundary map $\delta_* : L_{n-2}H_{2(n-2)+1}(U) \to L_{n-2}H_{2n}(D)$ must be zero. Hence a comes from an element $b \in L_{n-2}H_{2(n-2)+1}(\tilde{X}_Y)$. If $\bar{b} := \Phi_{n-2,2(n-2)+1}(b) \neq 0$, then $\exists c \in L_{n-2}H_{2(n-2)+1}(D)$ such that $b - i_*(c) \in L_{n-2}H_{2(n-2)+1}(\tilde{X}_Y)_{hom}$ and $j^*(b-i_*(c)) = a$. In fact, since $j^*(\bar{b}) = 0$, there exists $\bar{c} \in H_{2(n-2)+1}(D)$ such that $(i_0)_*(\bar{c}) = \bar{b}$. Note that $\Phi_{n-2,2(n-2)+1} : L_{n-2}H_{2(n-2)+1}(D) \to H_{2(n-2)+1}(D)$ is an isomorphism by Theorem 4.1, then there exists $c \in L_{n-2}H_{2(n-2)+1}(D)$ such that $\Phi_{n-2,2(n-2)+1}(c) = \bar{c}$. This shows the surjectivity in this case.

(c) Now we only need to consider the situation that $k \geq 2(n-2)+2$. In this case, the surjectivity of $j^* : L_{n-2}H_k(\tilde{X}_Y)_{hom} \to L_{n-2}H_k(U)_{hom}$ is from the commutative diagram in Lemma 2.2, and the surjectivity of the map $\Phi_{n-2,k} : L_{n-2}H_k(D) \to H_k(D)$ (see Theorem 4.1). In fact, if $a \in L_{n-2}H_k(U)_{hom}$, then by the exactness of the commutative diagram in Lemma 2.2, there is an element $b \in L_{n-2}H_k(\tilde{X}_Y)$ such that $j^*(b) = a$. Set $\bar{b} = \Phi_{n-2,k}(b)$. Since $j^*(\bar{b}) = 0 \in H_k^{BM}(U)$, $\exists \bar{c} \in H_k(D)$ such that $i_*(\bar{c}) = \bar{b}$. Now $\Phi_{n-2,k} : L_{n-2}H_k(D) \cong H_k(D)$ (See Theorem 4.1), there exists $c \in L_{n-2}H_k(D)$ such that $\Phi_{n-2,k}(c) = \bar{c}$. The commutative diagram in Lemma 2.2 implies that $\Phi_{n-2,k}(b - i_*(c)) = 0$, i.e., $b - i_*(c) \in L_{n-2}H_k(\tilde{X}_Y)_{hom}$. The exactness of the upper row in Lemma 2.2 gives $j^*(b - i_*(c)) = a$. This completes the surjectivity in this case.

This completes the proof for a blow-up along a smooth subvariety Y of codimension at least 2 in X.

Now recall the weak factorization Theorem proved in [AKMW] (and also [W]) as follows:

Theorem 4.2 *([AKMW] Theorem 0.1.1, [W]) Let $\varphi : X \to X'$ be a birational map of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as a sequence of birational maps*

$$X = X_0 \overset{\varphi_1}{\to} X_1 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{n+1}}{\to} X_n = X'$$

where each X_i is a smooth complete variety, and $\varphi_{i+1} : X_i \to X_{i+1}$ is either a blowing-up or a blowing-down of a smooth subvariety disjoint from U.

Note that $\varphi : X \to X'$ is birational between projective manifolds. We complete the proof of for the birational invariance of $L_{n-2}H_k(X)_{hom}$ for any smooth X by applying the above theorem.

$\square$

Remark 4.2 *Griffiths [G] showed the nontriviality of the Griffiths group of 1-cycles of general quintic hypersurfaces in* P^4 *and Friedlander [F] showed that* $L_1H_2(X)_{hom} \cong \mathrm{Griff}_1(X)$ *for any smooth projective variety* X*. Hence, in general, this is a* **nontrivial** *birational invariant even for projective threefolds.*

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